

[Second order P.D.E. with  
Two variable coefficients]

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Monge's Method: - Let the equation be

$$Rr + Ss + Tt = V \dots \dots \dots (1)$$

where  $R, S, T$  and  $V$  are fns. of  $x, y, z$  and  $p, q$   
 $r, s, t$  have their usual meanings.

i.e.  $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \cdot \partial y},$

$$t = \frac{\partial^2 z}{\partial y^2}$$

Now,  $r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial p}{\partial x}$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial q}{\partial y}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial q}{\partial x}$$

and also  $s = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial p}{\partial y}$

$\therefore$  we have

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy \Rightarrow$$

$$dp = rdx + sdy$$

$$\Rightarrow r = \frac{dp - sdy}{dx}$$

and  $dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = sdx + tdy$

$$\Rightarrow t = \frac{dq - sdx}{dy}$$

Putting these values of  $r$  and  $t$  in (1), we get

$$R \left( \frac{dp - sdy}{dx} \right) + Ss + T \left( \frac{dq - sdx}{dy} \right) = V$$

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$$\Rightarrow (R dp dy + T dq dx - V dxdy) -$$

$$s(R dy^2 - S dxdy + T dx^2) = 0 \dots \dots \dots (2)$$

Clearly, any relation between  $x, y, z, p, q$  which satisfies (2) must also satisfy the following two simultaneous equations.

$$R dp dy + T dq dx - V dxdy = 0 \dots \dots \dots (3)$$

$$\text{and } R dy^2 - S dxdy + T dx^2 = 0 \dots \dots \dots (4)$$

Equations (3) and (4) are called Monge's subsidiary equations.

Equation (4) being a quadratic in general, it can be resolved into two equations, say

$$dy - m_1 dx = 0 \text{ and}$$

$$dy - m_2 dx = 0 \dots \dots \dots \dots \dots (5)$$

**Case I:** — When  $m_1$  and  $m_2$  are distinct (distinct-DIST) in (5). In this case  $dy - m_1 dx = 0$  and (3) if necessary by using well-known result  $dz = pdx + q dy$ , will give two integrals  $u_1 = a$  and  $v_1 = b$ .

$$\text{Then the relation } u_1 = f_1(v_1) \dots \dots \dots (6)$$

is the solution of and is called Intermediate Integral. Similarly from (4) and  $dy - m_2 dx = 0$  obtain another Intermediate Integral

$$u_2 = f_2(v_2) \dots \dots \dots \dots \dots (7)$$

From (6) and (7) we find the values of  $p$  and  $q$  in terms of  $x$  and  $y$ .

Substituting these values of  $p$  and  $q$  in

$dz = pdx + qdy$  and integrating it,  
the complete integral is obtained.

Case II:— when  $m_1 = m_2$  i.e. (4) is a perfect square. In this case we get only one Intermediate Integral which is of the form

$$Pp + Qq = R.$$

This is solved with the help of Lagrange's method.

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Q. ① Solve:  $r + (a+b)s + abt = xy$

Solution:— We have,

$$\begin{aligned} dp &= sdx + dy \quad \text{and} \\ dq &= sdx + tdy. \end{aligned}$$

These give  $s = \frac{dp - dy}{dx}$  and  $t = \frac{dq - dx}{dy}$

Putting these values of  $s$  and  $t$  in the given equation, we get

$$\frac{dp - dy}{dx} + (a+b)s + ab \cdot \frac{dq - dx}{dy} = xy$$

$$\Rightarrow dpdy - dy^2 + (a+b)sdx dy + ab(dqdx - dx^2) = xydxdy$$

$$\Rightarrow (dpdy + abdqdx - xydxdy)$$

$$- s\{dy^2 - (a+b)dxdy + abdx^2\} = 0$$

Then, Monge's subsidiary equations are

$$dpdy + abdqdx - xydxdy = 0 \dots \dots \dots (1)$$

$$\text{and } dy^2 - (a+b)dxdy + abdx^2 = 0 \dots \dots \dots (2)$$

Two factors of (2) are

$dy - adx = 0$ , Integratief

and  $dy - dx = 0$ , Integrating

Combining (3) with (1), we get

$$ad\varphi + abdd\varphi - ax(c_1 + ax)dx = 0$$

$$\text{i.e. } dp + bddx - x(c_1 + ax)dx = 0$$

## Integration,

$$p + bq - \left( c_1 \frac{x^2}{2} + a \frac{x^3}{3} \right) = A$$

$$\text{i.e. } p + b\alpha + \frac{1}{2} \alpha^2 (\gamma - \alpha x) - \frac{1}{3} \alpha x^3 = f_1(\gamma - \alpha x)$$

$$\text{or, } p + by + \frac{1}{6} \alpha x^3 - \frac{1}{2} x^2 y = f_1(y - \alpha x) \dots \dots \dots (5)$$

where  $f_i$  is an arbitrary function.

Similarly (replacing  $a$  by  $b$  and  $b$  by  $a$ ), the other intermediate integral obtained by combining (4) and (1) is

$$p + \alpha q + \frac{1}{6} b x^3 - \frac{1}{2} x^2 y = f_2(y - bx) \quad \dots \dots \dots (6)$$

where  $f_2$  is also arbitrary function.

Now, (5) and (6) give

$$P = \frac{1}{2} y x^2 - \frac{1}{6} (a+b)x^3 \left[ \frac{1}{b-a} \right] \left[ a f_1(y-ax) - b f_2(y-bx) \right]$$

$$\text{and } \varphi = \frac{1}{6}x^3 + \left[ \frac{1}{b-a} \right] [f_1(y - ax) - f_2(y - bx)]$$

Putting these values in the equations

$$dz = \frac{1}{2}x^2ydx + \frac{1}{6}x^3dy - (a+b)\frac{1}{6}x^3dx$$

$$+ \frac{1}{b-a} [f_1(y-a\alpha)(dy - adx) - f_2(y-b\alpha)(dy - bdx)]$$

Integrating it the complete solution of the given diff. equation is

$$z = \frac{1}{6}x^3y - (a+b)\frac{x^2}{24} + F_1(y-ax) + F_2(y-bx)$$

where  $F_1$  and  $F_2$  are arbitrary functions.

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